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Industrial & Systems
Engineering Department
University of Florida
Gainesville, FL. 32611

OPTIMAL CONTROL OF MULTI-SHOP SYSTEMS
PART I: PARALLEL SHOPS
PART II: SERIES SHOPS

Research Report No. 78-4

by

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April, 1978

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In this paper we considers the optimal control structure for multi-shop (Part I: Parallel, Part II: Series) systems, where the input to the shop system is random and the shop output is determined by the number of workers in the shop. The number of workers available to the system is held constant, while control is exercised in discrete time by adjusting the allocation of workers to the various shops in the system. There is a cost for transferring workers. Additionally, there is a cost of holding backlog in the system. The control objective is to minimize the sum of these costs over an infinite horizon.

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20. Abstract (cont'd)

It is shown that for some "regions" of the system's state space the optimal control policies are known exactly without resorting to computational methods. For other "regions" it is shown that the problem can be decomposed into sub-problems of reduced complexity. Finally, an inertia (hysteresis) property is established which reduces the number of policy combinations which must be considered in some cases, and completely eliminates the necessity to determine policy in other cases. The net result is a substantial reduction in the computer storage and computational effort required to solve for the optimal control policy.

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Abstract

In this paper we consider the optimal control structure for multi-shop (Part I: Parallel, Part II: Series) systems, where the input to the shop system is random and the shop output is determined by the number of workers in the shop. The number of workers available to the system is held constant, while control is exercised in discrete time by adjusting the allocation of workers to the various shops in the system. There is a cost for transferring workers. Additionally, there is a cost of holding backlog in the system. The control objective is to minimize the sum of these costs over an infinite horizon.

It is shown that for some "regions" of the system's state space the optimal control policies are known exactly without resorting to computational methods. For other "regions" it is shown that the problem can be decomposed into subproblems of reduced complexity. Finally, an inertia (hysteresis) property is established which reduces the number of policy combinations which must be considered in some cases, and completely eliminates the necessity to determine policy in other cases. The net result is a substantial reduction in the computer storage and computational effort required to solve for the optimal control policy.

PART I

Introduction

Consider a multi-shop system. Work (measured in man-hours) arrives at the system periodically, say once a week. The amount of work arriving, to each shop per period is assumed to be random variable which can be described by a stationary probability density function. The amount of work processed out of a shop during a period is proportional to the number of workers in the shop. The number of workers in the system is constant. Control can be exercised by adjusting the allocation of workers to the various shops in the system.

A multitude of paper's have dealt with the characteristics of shops (queues) under output control. Howard [14], Wolfe and Dantzig [26], and Manne [18] developed numerical techniques to establish optimal control policies for Markov processes. The method, although general, established the basic analytic structure for determining optimal control policies for Markov queues with variable servers. Many other researchers, including Heyman [13], Sobel [24], Yadin and Naor [27], Magazine [16], [17], McGill [19], Crabill [7], [8], Beja [3], Blackburn [6], Balachandran [1], [2], Rata [22], Reed [23], Faddy [10], [11], Deb [9], Bell [4], [5], Winston [25], Odkenyi [20], [21], and Lippman [15] have contributed to the field. The shop system considered here is unique in that man power (i.e., a server) is treated as a shared resource available to several shops (queues).

The motivation for this study comes from observations of a Naval Air Rework Facility (NARF). A NARF is a large job shop devoted to depot level repair of naval aircraft and aircraft spare parts. NARFs maintain both extensive work standards on all jobs performed and extensive data systems which pinpoint each job's location in the shop system. As a consequence,

at any time, the amount of work backlog in any shop (in man-hours) is known. Control of the shop backlogs can be affected by transferring workers between shops. This is generally easier than authorizing overtime from managements' point of view.

There is a cost of maintaining backlog in the system which is roughly proportional to the amount of backlog. The backlog in the NARF shops can't be used by the fleet as spare parts. As a consequence, the higher the NARF shop backlogs, the higher the Navy's investment in spares must be. There is also a cost (primarily administrative) of transferring a worker from one shop to another. Clearly, the act of transferring workers implies that the worker involved has the prerequisite skills for that shop to which he is transferred. It follows that a NARF, for control purposes, can be broken into sub-systems of shops with similar worker skills.

There is no claim here that this description of the shop control problem of a NARF takes into account all facets of what is an extremely complex real-life control problem. It does, however, capture the essence of the system. In this paper, an analytic model of a shop system is developed. The system is modeled as a discrete time Markov-Decision process. Characteristics of the optimal control structure for the system are derived.

In Part 1, we approach the parallel shop case. In Part 2, the series shop case is considered. In both cases, the development is for two shop systems. The extensions to N-shops are straight-forward, but lengthy. The interested reader is referred to Haas [12] which is available from the authors.

The Two Shop System (Parallel)

Let us now address the problem of determining general properties for the optimal control structure of a system with 2 parallel shops which are controlled by transferring workers from one shop to the other.

At equally spaced points in time (surge points) a quantity of work may enter the shop. These quantities, measured in work units, are independently, but not necessarily identically, distributed random variables. Work which cannot immediately be processed is held as backlog for the shop it entered. The transfer of backlog from one shop to another is prohibited. In addition, the flow of work through each shop does not pass through the other shop. Thus, the system may be thought of as two shops in parallel (See Figure 1).

The total number of workers in the system is W. A worker may be transferred at specific points in time (those points to be defined shortly) from one shop to the other at a cost of T. Each of the W workers (while working in shop i) produce R_i work units over an interval of time between two consecutive input surges. Note that the couching of input in terms of 'work units' rather than 'man-hours' and using an efficiency rating of R_1 allows for differences in shop operational characteristics.

There is a backlog holding cost of H per unit incurred immediately before a surge point to the shops. So, if B_i is the backlog in shop i immediately before input surge n, the total backlog holding cost in period n is $H(B_1 + B_2)$

Control of the system is exercised by transferring workers from shop to shop immediately after each surge point (see Figure 2 where the transfer of a worker from one shop to the other results in a change in the slope of the backlog function). The objective is to exercise control so as to minimize the total of holding and transfer costs over some horizon.

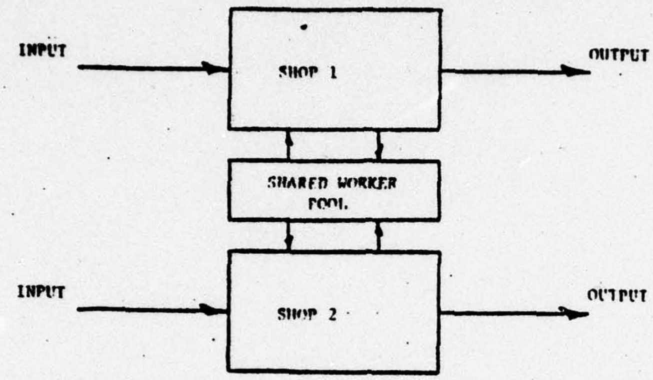

Figure 1. Two shops in parallel.

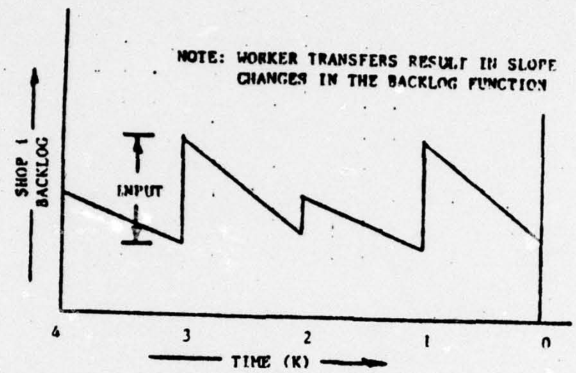

Figure 2. Realization of backlog function.

If the process is imbedded immediately after each surge point, it can be formulated as a Markov-Decision process. In order to formulate the optimization as a dynamic program, the state of the system at stage K is defined as a four dimensional vector $(\bar{B}, \bar{d})$. This vector may also be written as (B_1, B_2, d_1, d_2), where B_i is the backlog in shop i at stage K and d_i is the number of men in shop i at stage K + 1. It is easy to convince one's self that the state definition satisfies the Markov property. Before continuing, the following two definitions are required:

$\phi(\bar{B}, \bar{d})$ = the minimum expected discounted cost over an infinite horizon, given that there are now $\bar{B}$ units in the backlogs and there were $\bar{d}$ men in the shops during the previous transition $(0 \leq B_i < \infty,\ d_i = 0, 1, 2, \ldots, W,\ i = 1, 2)$.

$f^{\bar{d}}(\bar{m}|\bar{B})\, d\bar{m}$ = the probability that the backlog at the end of the transition is between $\bar{m}$ and $\bar{m} + d\bar{m}$ given that backlog at the beginning of the transition is $\bar{B}$ and, that policy $\bar{d}$ is in effect during the transition $(0 \leq m_i < \infty)$.

Let T be the cost of transferring a worker from one shop to another. Let $\bar{p} = (p_1, p_2)$ be the present worker allocation and $\bar{d} = (d_1, d_2)$ be the worker allocation during the previous transition. The recursive equation can now be written. It includes the transfer cost and backlog holding cost for the immediate transition plus the discounted expected cost over an infinite horizon. Thus,

$$\phi(\bar{B}, \bar{d}) = \min_{0 \leq p_1 \leq W} \{T\,|p_1 - d_1| + H \sum_{i=1}^{2} B_i + \alpha \int_0^{\infty} f^{\bar{p}}(\bar{m}|\bar{B})\ \phi(\bar{m}, \bar{p})\, d\bar{m}\}.$$

(Note: $0<\alpha<1$ is the discount factor)

The Structure of the Optimal Control Policy

In this section four fundamental properties of the optimal control policies for an infinite horizon will be determined. These properties will allow one to determine a great deal about the structure of the optimal control policy.

Property 1:

The first property, stated and proved in the theorem below, shows that when shop backlogs are high enough the optimal policy is known immediately. Before beginning the proof for property 1, two necessary lemmas will be established.

Lemma 1

$$\Phi(B_1 + \theta, B_2, \bar{d}) - \Phi(B_1, B_2, \bar{d}) \le \frac{\theta H}{1 - \alpha} .$$

(The same property holds in B_2.)

Proof:

Take two starting points in backlog space,

$$\text{Point 1} = (B_1, B_2, \bar{d}) \text{ and Point 2} = (B_1 + \theta, B_2, \bar{d}).$$

For both points take a specific input realization. For that input realization and starting Point 1, there are a set of optimal policies which will be enforced at each stage. For the same input realization, assume that these policies will be enforced for the sequence of points starting at Point 2. Then for each stage the separation between the sequence of points starting at Point 1 and those starting at Point 2 is no more than θ. Thus, differences in holding costs at each stage are no more than $H\theta$. Transfer costs are equal at each stage. For the infitite horizon this implies that the total difference in cost must be less than or equal to,

$$H \cdot \sum_{i=0}^{\infty} \alpha^i \theta = \frac{H \cdot \theta}{1 - \alpha}$$

The actual expected cost under optimal control must be less than or equal to this. Thus the lemma must hold (it should be noted that in the limit (as $B_1 \to \infty$) it is easy to see that the lemma inequality becomes an equality).

Q.E.D.

<u>Lemma 2</u>:

$$|\Phi(\bar{B}, \bar{d}) - \Phi(\bar{B}, \bar{d}')| \leq |d_1 - d_1'|T$$

<u>Proof</u>:

Let

$$\Psi(\bar{B}, \bar{p}) = \alpha \iint f^{\bar{p}}(\bar{m}|\bar{B}) \, \Phi(\bar{m}, \bar{p}) d\bar{m}$$

Let $\bar{p}$ be the optimal policy for $(\bar{B}, \bar{d})$. If $\bar{p}$ is made the policy for $(\bar{B}, \bar{d}')$, then the expected discounted return for that state is just,

$$\begin{aligned} \Phi^{\bar{p}}(\bar{B}, \bar{d}') &= T|p_1 - d_1'| + H(B_1 + B_2) + \Psi(\bar{B}, \bar{p}) \\ &\leq T|p_1 - d_1| + T|d_1 - d_1'| + H(B_1 + B_2) + \Psi(\bar{B}, \bar{p}) \\ &\leq T|d_1 - d_1'| + \Phi(\bar{B}, \bar{d}). \end{aligned}$$

The superscript, $\bar{p}$, on $\Phi^{\bar{p}}(\bar{B}, \bar{d}')$ indicates that $\bar{p}$, the policy for which the return is calculated, may not be optimal at $(\bar{B}, \bar{d}')$. Rearrangement of the above gives,

$$T|d_1 - d_1'| \geq \Phi^{\bar{p}}(\bar{B}, \bar{d}') - \Phi(\bar{B}, \bar{d}).$$

since

$$\Phi^{\bar{p}}(\bar{B}, \bar{d}') \geq \Phi(\bar{B}, \bar{d}'),$$

this implies that,

$$T|d_1 - d_1'| \geq \Phi(\bar{B}, \bar{d}') - \Phi(\bar{B}, \bar{d})$$

A similar argument shows that,

$$T|d_1' - d_1| \geq \Phi(\bar{B}, \bar{d}) - \Phi(\bar{B}, \bar{d}')$$

Hence,

$$T|d_1' - d_1| \geq | \Phi(B, d) - \Phi(B, d')|.$$

Q.E.D.

With the foregoing results established, one may prove the existence of property 1 for the infinite horizon.

Theorem 1: Let $R_2 > R_1$. Define

$$n = n_1 + n_2, \text{ and}$$

(1a) $$\theta = [T + n_1R_2H/(1 - \alpha)]\alpha^{n_1}.$$

1. When

(1b) $$T < H(R_2 - R_1)\alpha/(1 - \alpha)^2,$$

if n_2 is selected such that

(1c) $$\frac{R_2 - R_1}{2R_2 - R_1} \geq \alpha^{n_2}$$

(1d) $$\frac{H(R_2 - R_1)\alpha(1 - \alpha^{n_2+1})}{(1 - \alpha)^2} - T \geq \frac{\alpha^{n_2} R_2H}{(1 - \alpha)^2}, \text{ and}$$

(1e) $$T < H(R_2 - R_1) \sum_{j=1}^{n_2-1} j\alpha^j.$$

Then when

(1f) $$T \leq H(R_2 - R_1) \sum_{j=1}^{n_2-1} j\alpha^j - \theta, \text{ and}$$

$B_2 \geq n \cdot R_2 \cdot W$, it is optimal to place all workers in shop 2 for the next n_1 stages.

2. When

(1g) $$T > H(R_2 - R_1)\alpha\,/(1 - \alpha)^2 \quad ,$$

if n_2 is selected such that

(1c) $$\frac{R_2 - R_1}{2R_2 - R_1} \geq \alpha^{n_2} \quad \text{, and}$$

(1h) $$T - \frac{H(R_2 - R_1)\alpha(1 - \alpha^{n_2+1})}{(1 - \alpha)^2} \geq \frac{\alpha^{n_2}\, 2R_2H}{(1 - \alpha)^2} \; .$$

Then when

(1i) $$T \geq H(R_2 - R_1)\alpha/(1 - \alpha)^2 + \theta \text{ , and}$$

$B_i \geq nR_iW$ for all i, it is optimal to maintain the worker allocation from the previous stage for the next n_1 stages.

In other words when

$$T > H(R_2 - R_1)\alpha/(1 - \alpha)^2$$

There will always be an n such that there is a region $B_i \geq n \cdot R_i \cdot W$, for all i where it is optimal to maintain the worker allocation from the previous stage.

Furthermore, since as $n_2 \to \infty$ the term in (1f), $\sum_{j=1}^{n_2-1} j\alpha^j \to \frac{\alpha}{(1 - \alpha)^2}$, when

$$T < H(R_2 - R_1)\alpha(1 - \alpha)^2$$

there will always be an n large enough that there is a region $B_2 \geq n \cdot R_2 \cdot W$ where it is optimal to place all workers in shop 2.

Proof:

The method for completing the proof will be as follows:

It will first be assumed that it is never optimal in the first n_1 stages to decrease the number of workers in shop 2, and that for the first n_1 stages it is only optimal to increase the number of workers in shop 2 at the first stage.

In part 1, it will be shown that given the above assumption and given that (li) holds it is always optimal to maintain the number of workers in the previous stage. In part 2 (also given the above assumption), the optimality of a complete transfer of workers to shop 2 will be shown for the case where (lf) holds. In part 3, the proof will be completed by demonstrating that the initial assumption must hold.

Part 1: Assume that inequality (li) holds. Fix a specific input realization. The expected cost when no policy change is made for n_1 transitions is

$$(2) \qquad \Phi'(\bar{B}, \bar{p}) = C^h_{n_1} + \alpha^{n_1} C_{n_1+1}$$

where $C^h_{n_1}$ is expected discounted holding cost up through n_1 transitions and C_{n_1+1} is the expected discounted cost (under optimal control) after n_1 transitions. By assumption, there is no transfer cost, $C^t_{n_1}$, through the first n_1 transitions.

The expected cost (when a transfer of X workers from shop 1 to shop 2 is made at the current stage, the allocation is held constant over n_1 stages, and the system is optimally controlled thereafter) is,

(3) $$\Phi''(\bar{B}, \bar{p}) = C^{t'}_{n_1} + C^{h'}_{n_1} + \alpha^{n_1} C'_{n_1+1}.$$

where $c^{t'}_{n_1}$ is the expected discounted transfer cost through n_1 transitions

$C^{h'}_{n_1}$ is the expected discounted holding cost through n_1 transitions, and

C'_{n_1+1} is the expected discounted cost (under optimal control) after n_1 transitions.

For any input realization there can be no more than $n_1 \times R_i$ separating the individual shop backlogs for one control policy $(\bar{p})$ versus the other $(\bar{p}')$ after n_1 stages of the process. Thus,

$$|C_{n_1+1} - C'_{n_1+1}| = |\Phi(B_1 + \theta_1, B_2 - \theta_2, \bar{p}) - \Phi(B_1, B_2, \bar{p}')|$$

$$\leq |\Phi(B_1 + \theta_1, B_2 - \theta_2, \bar{p}) - \Phi(B_1, B_2, \bar{p})|$$

$$+ |\Phi(B_1, B_2, \bar{p}) - \Phi(B_1, B_2, \bar{p}')|,$$

$$\leq \theta_1 \leq n_1XR_1$$

$$\leq \theta_2 \leq n_1XR_2$$

Using Lemma 1, it is easily shown that

(5) $$|\Phi(B_1 + \theta_1, B_2 - \theta_2, \bar{p}) - \Phi(B_1, B_2, \bar{p})| \leq \frac{n_1R_2XH}{1 - \alpha}.$$

It is also known from Lemma 2 that,

(6) $$|\Phi(B_1, B_2, \bar{p}) - \Phi(B_1, B_2, \bar{p}')| \leq |p_1 - p_1'|T \leq XT$$

Thus from (4), (5), and (6)

(7) $$|C_{n_1+1} - C'_{n_1+1}| \leq \frac{n_1R_2XH}{1 - \alpha} + XT$$

Rearranging (1a)

$$(8) \qquad T = \frac{-n_1 R_2 H}{1-\alpha} + \frac{\theta}{\alpha^{n_1}}$$

Substituting (8) into (7),

$$C_{n_1+1} - C'_{n_1+1} \leq \frac{X\theta}{\alpha^{n_1}}$$

Removing the absolute value sign and multiplying by α^{n_1} gives,

$$(9) \qquad \alpha^{n_1} C_{n_1+1} \leq X\theta + \alpha^{n_1} C'_{n_1+1}$$

Now consider the relationship of $C^h_{n_1}$ and $C^{h'}_{n_1}$. Since neither shop can run out of work, $C^h_{n_1}$ is $XH(R_2 - R_1) \sum_{j=1}^{n_1-1} j\alpha^j$ greater than $C^{h'}_{n_1}$. Thus

$$(10) \qquad C^h_{n_1} = C^{h'}_{n_1} + XH(R_2 - R_1) \sum_{j=1}^{n_1-1} j\alpha^j$$

Adding (9) and (10) together yields,

$$C^h_{n_1} + \alpha^{n_1} C_{n_1+1} \leq X(\theta + H(R_2 - R_1) \sum_{j=1}^{n_1-1} j\alpha^j) + C^{h'}_{n_1} + \alpha^{n_1} C'_{n_1+1}$$

Using inequality (1i) and noting that $\sum_{j=1}^{\infty} j\alpha^j = \alpha/(1-\alpha)^2$,

$$(11) \qquad C^h_{n_1} + \alpha^{n_1} C_{n_1+1} \leq XT + C^{h'}_{n_1} + \alpha^{n_1} C'_{n_1+1}.$$

But since $C_{n_1}^{t'} = XT$ it is immediate from (11) that,

$$C_{n_1}^{h} + \alpha^{n_1} C_{n_1+1} \leq C_{n_1}^{t'} + C_{n_1}^{h'} + \alpha^{n_1} C'_{n_1+1},$$

or simply

$$\Phi'(\bar{B}, \bar{d}) \leq \Phi''(\bar{B}, \bar{d}).$$

<u>Part 2</u>: Assume that inequality (1f) holds.

For this part of the theorem the definitions for (2) and (3) still hold. The argument given in part 1 showed that

$$|C_{n_1+1} - C'_{n_1+1}| \leq \frac{X\theta}{\alpha^{n_1}}$$

Thus,

$$\text{(12)} \qquad \alpha^{n_1} C'_{n_1+1} \leq \theta X + \alpha^{n_1} C_{n_1+1}.$$

Since shop 2 cannot run out of work through n_1 transitions,

$$\text{(13)} \qquad C_{n_1}^{h'} \leq C_{n_1}^{h} - XH(R_2 - R_1) \sum_{j=1}^{n_1-1} j\alpha^j$$

(Note: if shop 1 does not run out of work during the first n_1 stages, the above inequality becomes an equality.)

Adding (12) and (13) gives

$$C_{n_1}^{h'} + \alpha^{n_1} C'_{n_1+1} \leq X(\theta - H(R_2 - R_1) \sum_{j=1}^{n_1-1} j\alpha^j) + C_{n_1}^{h} + \alpha^{n_1} C_{n_1+1}.$$

but (1f) requires that:

$$T \leq H(R_2 - R_1) \sum_{j=1}^{n_1-1} j\alpha^j - \theta$$

Thus,

$$XT + C^{h'}_{n_1} + \alpha^{n_1} C'_{n_1+1} \leq C^{h}_{n_1} + \alpha^{n_1} C_{n_1+1}$$

or simply,

$$\Phi''(\bar{B}, \bar{d}) \leq \Phi'(\bar{B}, \bar{d}).$$

Part 3: The proof is completed by showing that the optimal policy at stage 1 will remain optimal through the first n_1 stages and by showing that it is never optimal to decrease the number of workers in shop 2. This will be shown by considering two cases separately.

Case 1: In case 1 it will be shown that when the conditions in the proof statement hold it is never optimal in the first n_1 stages to decrease the number of workers in shop 2.

Let the optimal policy at stage Z be to transfer X workers from shop 2 to shop 1. If $\bar{d} = (d_1, d_2)$ is the policy of the previous stage, the policy at stage Z is $(d_1 + X, d_2 - X)$. The system is optimally controlled thereafter. The shop system with this control policy will hereafter be designated option 1.

Consider a second system which is identical to the first except that the policy in force at stage Z is (d_1, d_2). All policies before and after stage Z are identical to those of option 1. Call this second system, option 2.

Now fix an input realization and compare the expected return for the two options. It will be shown that option 2 provides the lowest expected return, and therefore that the postulated decrease is never optimal.

For option 2, since shop 2 will not run out of work for at least the first $n_1 + n_2$ stages, the holding cost is at least

$$XH(R_2 - R_1) \sum_{j=1}^{n_1+n_2-Z} \alpha^{j+Z-1}$$

less than for option 1. Transfer costs for option 2 are less than or equal to those for option 1. Thus,

$$(14) \qquad C^{h'}_{n_1+n_2} + C^{t'}_{n_1+n_2} \leq C^{h}_{n_1+n_2} + C^{t}_{n_1+n_2} - XR(R_2 - R_1) \sum_{j=1}^{n_1+n_2-Z} \alpha^{j+Z-1}$$

where

$C^{h'}_{n_1+n_2}$ = Option 2 holding cost for the first $n_1+ n_2$ stages

$C^{t'}_{n_1+n_2}$ = Option 2 transfer cost for the first n_1+n_2 stages

$C^{h}_{n_1+n_2}$ = Option 1 holding cost for the first n_1+n_2 stages

$C^{t}_{n_1+n_2}$ = Option 1 transfer cost for the first n_1+n_2 stages

At the end of n_1+n_2 stages if the system state for option 1 is $(B_1, B_2, \bar{d})$, the system state for option 2 will be $(B_1 - \theta R_1\ B_2 + XR_2, \bar{d})$, $\theta \leq X$. Lemma (1) can be used to show that the maximum possible difference in expected cost for the ensuing stages will be

$$\frac{\alpha^{n_1+n_2}}{1-\alpha} XR_2H.$$

Thus if

$C'_{n_1+n_2+1}$ = Option 2 total expected cost for stages n_1+n_2+1 on

$C_{n_1+n_2+1}$ = Option 1 total expected cost for stages n_1+n_2+1 on

$$(15) \qquad C'_{n_1+n_2+1} \leq C_{n_1+n_2+1} + \frac{\alpha^{n_1+n_2} XR_2H}{1-\alpha}.$$

Combining (14) and (15) gives

(16) $$C^{h'}_{n_1+n_2} + C^{t'}_{n_1+n_2} + C'_{n_1+n_2+1} \leq C^{h}_{n_1+n_2} + C^{t}_{n_1+n_2} + C_{n_1+n_2+1}$$

$$- XH(R_2 - R_1) \sum_{j=1}^{n_1+n_2-Z} \alpha^{j+Z-1} + \frac{\alpha^{n_1+n_2}}{1-\alpha} XR_2H.$$

Now from (16), so long as

$$0 \geq \frac{\alpha^{n_1+n_2}}{1-\alpha} XR_2H - XH(R_2 - R_1) \sum_{j=1}^{n_1+n_2-Z} \alpha^{j+Z-1},$$

total option 2 expected cost will be less than that for option 1. Certainly this will be true for any policy changes occuring in the first n_1 stages if

$$0 \geq \frac{\alpha^{n_1+n_2}}{1-\alpha} XR_2H - XH(R_2 - R_1) \sum_{j=1}^{n_2} \alpha^{j+n_1-1}.$$

Or simplifying if,

$$0 \geq \frac{\alpha^{n_2}}{1-\alpha} R_2 - (R_2 - R_1) \sum_{j=1}^{n_2} \alpha^{j-1}.$$

But this is equivalent to condition (1c) in the theorem statement.

Case 2: Now to conlude the proof it will be shown that it is never optimal to increase the number of workers in shop 2 at any but the first stage. Assume that at some stage Z (not the first stage) the policy is to increase the number of workers in shop 2 by X. So that if $\bar{d} = (d_1, d_2)$ is the policy of the previous stage the policy at the current stage is $(d_1\text{-}X, d_2\text{+}X)$. The system is optimally controlled thereafter. The shop with this control policy will be designated option 1.

Consider a second system with a policy at the previous stage, Z-1, of $(d_1\text{-}X, d_2\text{+}X)$ and policies which are the same as option 1 for all stages before and after Z-1. Now fix an input realization and compare the expected return

for the two options. Let the system costs for this case take the same symbols as those for costs in Case 1.

It is immediately clear that

$$(17)\qquad C^{t'}_{n_1+n_2} = C^{t}_{n_1+n_2} + X\alpha^{Z-2}\,(1-\alpha)T.$$

Furthermore,

$$(18)\qquad C^{h'}_{n_1+n_2} \le C^{h}_{n_1+n_2} - XH(R_2 - R_1)\sum_{j=1}^{n_1+n_2-Z+1} \alpha^{j+Z-2}$$

Also,

$$(19)\qquad C'_{n_1+n_2+1} \le C_{n_1+n_2+1} + \frac{\alpha^{n_1+n_2}}{1-\alpha}\, XR_2H$$

Adding (17), (18), and (19) gives

$$(20)\qquad C^{h'}_{n_1+n_2} + C^{t'}_{n_1+n_2} + C'_{n_1+n_2+1} \le C^{h}_{n_1+n_2} + C^{t}_{n_1+n_2} + C_{n_1+n_2+1}$$

$$+ X\alpha^{Z-2}\,(1-\alpha)T + \frac{\alpha^{n_1+n_2}}{1-\alpha}\, XR_2H - XR(R_2 - R_1)\sum_{j=1}^{n_1+n_2-Z+1} \alpha^{j+Z-2}$$

Now from (20), so long as

$$(21)\qquad 0 \ge \alpha^{Z-2}(1-\alpha)T + \frac{\alpha^{n_1+n_2}}{1-\alpha}\, R_2H - H(R_2 - R_1)\sum_{j=1}^{n_1+n_2-Z+1} \alpha^{j+Z-2},$$

total option 2 expected cost will be less than that for option 1.

Inequality (21) may be rearranged to read

$$0 \ge \alpha^{Z-2}\left[(1-\alpha)T - H(R_2 - R_1)\sum_{j=1}^{n_1+n_2-Z+1} \alpha^{j}\right] + \frac{\alpha^{n_1+n_2}}{1-\alpha}\, R_2H$$

Note that

$$\alpha^{Z-2}\left[(1-\alpha)T - H(R_2 - R_1)\sum_{j=1}^{n_1+n_2-Z+1}\alpha^j\right] + \frac{\alpha^{n_1+n_2}}{1-\alpha} R_2H$$

$$\leq \alpha^{Z-2}\left[(1-\alpha)T - H(R_2 - R_1)\sum_{j=1}^{n_2+1}\alpha_1^j\right] + \frac{\alpha^{n_1+n_2}}{1-\alpha} R_2H.$$

Note from the conditions of the theorem statement (i.e., (1b) and (1c), when

$$(1-\alpha)T - H(R_2 - R_1)\sum_{j=1}^{\infty}\alpha^j < 0$$

n_2 must be selected so that

$$(1-\alpha)T - H(R_2 - R_1)\sum_{j=1}^{n_2+1}\alpha^j < 0.$$

Thus,

$$\alpha^{Z-2}\left[(1-\alpha)T - H(R_2 - R_1)\sum_{j=1}^{n_2+1}\alpha^j\right] + \frac{\alpha^{n_1+n_2}}{1-\alpha} R_2H$$

$$\leq \alpha^{n_1}\left[(1-\alpha)T - H(R_2 - R_1)\sum_{j=1}^{n_2+1}\alpha^j\right] + \frac{\alpha^{n_1+n_2}}{1-\alpha} R_2H.$$

To summarize, this argument shows that if n_1 and n_2 are selected so that (1b) and (1c) hold, then

$$\alpha^{Z-2}(1-\alpha)T + \frac{\alpha^{n_1+n_2}}{1-\alpha} RH - H(R_2 - R_1)\sum_{j=1}^{n_1+n_2-Z+1}\alpha^{j+Z-2}$$

$$\leq \alpha^{n_1}\left[(1-\alpha)T - H(R_2 - R_1)\sum_{j=1}^{n_2+1}\alpha^j\right] + \frac{\alpha^{n_1+n_2}}{1-\alpha} R_2H \leq 0$$

This means that option 2 expected cost is less than option 1 expected cost. Thus when (1b) and (1c) hold it is never optimal to increase the number of workers in shop 2 at any stage but 1 (or if it is optimal to increase the

number of workers at a stage other than 1, an alternative optimal policy exists at that stage which does not increase the number of workers in shop 2.)

When (1e) does not hold then of course,

$$T > H(R_2 - R_1)\alpha/(1 - \alpha)^2$$

In this case the nonoptimality of a shop 2 worker increase at any stage other than 1 can be shown with a similar argument. Here such an increase is postulated at stage Z. It can then be shown that when (1g) holds the expected cost is reduced by <u>delaying</u> the worker increase by one stage.

Q.E.D.

<u>Property 2</u>:

Just as backlog can be so large that an optimal policy at stage k is known immediately, backlog levels may be so low that the optimal policy is known immediately.

Before proving this property, it is necessary to note that the discounted system cost function is increasing in B_i.

<u>Lemma 3</u>:

For the two shop system;

$$\Phi(B_1 + \Theta, B_2, \bar{d}) - \Phi(B_1, B_2, \bar{d}) \geq H\Theta \text{ and}$$

$$\Phi(B_1, B_2 + \Theta, \bar{d}) - \Phi(B_1, B_2, \bar{d}) \geq H\Theta$$

Proof: The proof is obvious:

Theorem 2: If the state of the system $(\bar{B}, \bar{d})$

$= (B_1, B_2, d_1, d_2)$ is such that $B_1 \leq d_1 \cdot R_1$ and

$B_2 \leq d_2 \cdot R_2$, the optimal policy is to maintain the

manpower allocation of the previous stage, $\bar{d}$.

The above restriction on the state space merely requires that if the manpower allocation of the previous stage is maintained, before the next input surge both shops will become idle. Given the stated restrictions on the state space, a change in worker allocation can only increase the backlog in the shops. Thus, a change in policy is premature. It is better to wait until one knows where the system will be after the next input surge, and then, if necessary, make a policy change.

Proof: Suppose that the state of the system satisfies the requirements immediately above. Then, since $B_1 \leq d_1 \cdot R_1$ and $B_2 \leq d_2 \cdot R_2$,

$$f^{\bar{d}}(m_1, m_2 | B_1, B_2) = f^{\bar{d}}(m_1, m_2 | 0, 0).$$

From this and Lemma 2 it is clear that;

$$\iint \Phi(m_1, m_2, \bar{d})\, f^{\bar{P}}(m_1, m_2 | B_1, B_2)\, dm_1\, dm_2$$

$$\geq \iint \Phi(m_1, m_2, \bar{d})\, f^{\bar{d}}(m_1, m_2 | B_1, B_2)\, dm_1\, dm_2.$$

Also, from Lemma 3:

(23) $\quad T|p_1 - d_1| + \Phi(m_1, m_2, \bar{p}) \geq \Phi(m_1, m_2, \bar{d})$.

Substituting (23) into 22 and multiplying through by α gives

$$(24) \qquad \alpha T|p_1 - d_1| + \alpha\iint \Phi(m_1, m_2, \bar{p})\, f^{\bar{p}}(m_1, m_2|B_1, B_2)\, dm_1\, dm_2$$

$$\geq \alpha\iint \Phi(m_1, m_2, \bar{d})\, f^{\bar{d}}(m_1, m_2|B_1, B_2)\, dm_1\, dm_2$$

Inequality (24) is unchanged if $\alpha \leq 1$ is removed from the transfer term in the equation and $H(B_1 + B_2)$ is added to both sides. But the resulting expression is just an inequality relating the expected cost of the system under policy $\bar{p}$ and policy $\bar{d}$. Thus, $\bar{d}$ must be optimal.

Q.E.D

Property 3 (The Inertia Property)

The third property to be established is the inertia property for the two shop system. It is in a sense a generalization of Heyman's [12] main result to the two shop case. The inertia property for the two shop case is established by the following theorems:

Theorem 3: Consider two distinct points in the shop state space $(\bar{B}, \bar{d})$ and $(\bar{B}, \bar{d}')$. Let $\bar{p}$ be optimal at $(\bar{B}, \bar{d})$ and let $\bar{p}'$, be optimal at $(\bar{B}, \bar{d}')$. Furthermore, let $d_1 \geq d_1' \geq p_1$ and $d_1' \geq p_1'$ or let $d_1 \leq d_1' \leq p_1$ and $d_1' \leq p_1'$. Then $\bar{p}$ is also optimal at $(\bar{B}, \bar{d}')$, and $\bar{p}'$ is optimal at $(B, \bar{d})$.

Proof:

Once again, let

$$\Psi(\bar{B}, \bar{p}) = \alpha\iint f^{\bar{p}}(\bar{m}|\bar{B})\, \Phi(\bar{m}|\bar{p})d\bar{m}.$$

Since p is optimal at $(\bar{B}, \bar{d})$,

(25) $$H(B_1 + B_2) + T|d_1 - p_1| + \Psi(\bar{B}, \bar{p}) \leq H(B_1 + B_2) + T|d_1 - p_1'| + \Psi(\bar{B}, \bar{p}').$$

Likewise, since $\bar{p}'$ is optimal at $(\bar{B}, \bar{d}')$

(26) $$H(B_1 + B_2) + T|d_1' - p_1'| + \Psi(B, p') \leq H(B_1 + B_2) + T|d_1' - p_1| + \Psi(\bar{B}, \bar{p}).$$

Observe that when,

(27) $$d_1 \geq d_1' \geq p_1 \text{ and } d_1' \geq p_1'$$

$$|d_1' - p_1| = |d_1 - p_1| + (d_1' - d_1) \text{ and } |d_1' - p_1'| = |d_1 - p_1'| + (d_1' - d_1).$$

When

(28) $$d_1 \leq d_1' \leq p_1 \text{ and } d_1' \leq p_1'$$

$$|d_1' - p_1| = |d_1 - p_1| - (d_1' - d_1) \text{ and } |d_1' - p_1'| = |d_1 - p_1'| - (d_1' - d_1).$$

Since p_1 and p_1' must satisfy either equation (27) or (28), inequality (26) may be written,

(29) $$H(B_1 + B_2) + T|d_1 - p_1'| + \Psi(\bar{B}, \bar{p}') \leq H(B_1 + B_2) + T|d_1 - p_1| + \Psi(\bar{B}, \bar{p}).$$

Inequalities (25) and (29) imply that,

(30) $$H(B_1 + B_2) + T|d_1 - p_1'| + \Psi(\bar{B}, \bar{p}') = H(B_1 + B_2) + T|d_1 - p_1| + \Psi(\bar{B}, \bar{p}).$$

Since the left hand side of (33) is the expected return with a policy choice of $\bar{p}'$; since the right hand side is the expected return with $\bar{p}$ worker allocation, and since $\bar{p}$ is optimal at $(\bar{B}, \bar{d})$; $\bar{p}'$ must be optimal at $(\bar{B}, \bar{d})$ as well. The same argument can be used to show that $\bar{p}$ is optimal at $(\bar{B}, \bar{d})$.

Q.E.D.

Theorem 4: Fix some point $\bar{B}$ in backlog. Let $\bar{p}$ be the optimal policy when the number of workers in shop 1 at the previous transition is S. Let $\bar{p}'$ be the optimal policy at $\bar{B}$ when the number of workers in shop 1 at the previous transition is S + Q (Q > 0). Then,

$$p_1 \leq p_1'.$$

Proof:

The proof is executed by considering four possible cases,

Case 1: $p_1 \geq S$ and $p_1' \geq S + Q$

Case 2: $p_1 \leq S$ and $p_1' \leq S + Q$

Case 3: $p_1 \leq S$ and $p_1' \geq S + Q$

Case 4: $p_1 > S$ and $p_1' \leq S + Q$

The following arguments show that for each case $p_1 \leq p_1'$,

Case 1: If $S + Q \geq p_1$, then $p_1 < S + Q \leq p_1'$. If $S + Q \leq p_1$, then $p_1 = p_1'$ by Theorem 3.

Case 2: If $S < p_1'$ then $p_1 \leq S < p_1'$. If $S \geq p_1'$, then $p_1 = p_1'$ by theorem 3.

Case 3: $p_1 \leq S < S + Q \leq p_1'$

Case 4: For case 4 the fact that $p_1 \leq p_1'$ is shown by contradiction. Assume the opposite,

$$p_1 > p_1'$$

Since $\bar{p}$ is optimal at ($\bar{B}$, S, W - S),

$$\text{(33)} \qquad H(B_1 + B_2) + T|p_1 - S| + \Psi(\bar{B}, \bar{p}) \leq H(B_1 + B_2) + T|p_1' - S| + \Psi(\bar{B}, \bar{p}').$$

Since $\bar{p}'$ is optimal at ($\bar{B}$, S + Q, W - (S +Q))

$$(32) \quad H(B_1 + B_2) + T|p_1 - (S + Q)| + \Psi(\bar{B}, \bar{p}) \geq H(B_1 + B_2) + T|p_1' - (S + Q)| + \Psi(\bar{B}, \bar{p}').$$

With slight rearrangement and the elimination of $H(B_1 + B_2)$ on each side of the inequalities, (31) and (32) can be written as,

$$(33) \quad \Psi(\bar{B}, \bar{p}) \leq T|p_1' - S| - T|p_1 - S| + \Psi(\bar{B}, \bar{p}')$$

$$(34) \quad \Psi(\bar{B}, \bar{p}) \geq T|p_1' - (S + Q)| - T|p_1 - (S + Q)| + \Psi(\bar{B}, \bar{p}')$$

Subtracting (33) from (34) gives,

$$|p_1' - S| - |p_1 - S| \geq |p_1' - (S + Q)| - |p - (S + Q)|$$

But it is easily shown that for case 4

$$|p_1' - S| - |p_1 - S| < |p_1' - (S + Q)| - |p_1 - (S + Q)|.$$

Therefore a contradiction exists and p_1 must be less than or equal to p_1'.

Q.E.D.

Property 4:

When the backlog in one shop, say shop 2, becomes very large while the backlog in shop 1 is relatively low, certain simplifications in the control structure occur. The optimal policy structure for the system is sensitive only to the backlog level of the lightly loaded shop (shop 1). Consequently, the determination of optimal policies can be achieved by determining optimal policies for an equivalent single shop system. This result is described in the following theorem.

Theorem 4: As B_i becomes large, the optimal policy structure asymptotically approaches an optimal control structure which is defined by a single shop formulation. (The precise form for this single shop problem is given after some motivating development in the proof.)

Proof,

The recursive equation for the system is of course,

(35) $$\Phi(\bar{B}, \bar{d}) = \min_{0 \le d_1' \le W}\{T|d_1 - d_1'| + H(B_1 + B_2) + \alpha\int_0^\infty f^{\bar{d}'}(\bar{m}|\bar{B})\, \Phi(\bar{m}, \bar{d}')\, d\bar{m}\}.$$

Now consider the integral in detail. If Γ_2 is a random variable representing the input to the highly loaded shop (shop 2) during a transition then,

$$m_2 = B_2 + (d_1 - W) + \Gamma_2.$$

$$dm_2 = d\Gamma_2$$

Substituting these values into the integral in (35) yields,

(36) $$\int_0^\infty \bar{f}^{\bar{d}'}(m_1, B_2 + (d_1' - W) + \Gamma_2|\bar{B})\, \Phi(m_1, B_2 + (d_1' - W) + \Gamma_2, \bar{d}')\, dm_1\, d\Gamma_2$$

Once a worker allocation is selected for a particular period, the operation of the two shops for the period are independent, thus, the transition function may be broken into two parts. That is

$$\bar{f}^{d_1'}(m_1, B_2 + (d_1' - W) + \Gamma_2|\bar{B}) = g^{\bar{d}'}(m_1|B_1)\, f_2(\Gamma_2)$$

where

$f_i(\cdot)$ = the input distribution to shop i, and

$$g^{\bar{d}'}(m_1|B_1) = \int_0^{R_1 d_1' - B_1} f_1(\Gamma_1)\, d\Gamma_1, \text{ for } m_1 = 0$$

$$= f_1(m_1 + d_1' R_1 - B_1), \text{ for } m_1 > 0$$

Now (36) may be written,

(37) $$\int_0^\infty g^{d_1'}(m_1|B_1)\, f_2(\Gamma_2)\, \Phi(m_1, B_2 + (d_1' - W) + \Gamma_2, \bar{d}_1')\, dm_1\, d\Gamma_2.$$

In Lemma 1 it was noted that, for m_2 sufficiently large,

(38) $$\Phi(m_1, m_2 + \Theta\ \bar{d}') \cong \Phi(m_1, m_2, \bar{d}') + \frac{\Theta H}{1 - \alpha}.$$

Using (38), (37) may be rewritten as follows:

(39) $$\int_0^\infty g^{d_1'}(m_1|B_1)\ f_2(\Gamma_2)[\Phi(m_1, B_2, \bar{d}') + \frac{(d_1' - W + \Gamma_2)\ H}{1 - \alpha}]\ dm_1,\ d\Gamma_2.$$

Integrating through by Γ_2 and taking independent terms out, (39) becomes,

$$\int_0^\infty g^{d_1'}(m_1|B_1)\ \Phi(m_1, B_2, \bar{d}')\ dm_1 + \frac{d_1' - W)\ H}{1 - \alpha} + \frac{E(\Gamma_2)\ H}{1 - \alpha}.$$

Substituting (40) into (35) yields

(41) $$\Phi(\bar{B}, \bar{d}) = \min_{0 \le d_1' \le W} \{T|d_1 - d_1'| + H(B_1 + B_2) + \alpha \int_0^\infty g^{d_1'}(m_1|B_1)\ \Phi(m_1, B_2, \bar{d}')\ dm, + \frac{(d_1' - W)\ H}{1 - \alpha} + \frac{E(\Gamma_2)\ H}{1 - \alpha}\}.$$

Constants on the right hand side of (41) have no effect on the minimization process so that, insofar as policy determination is concerned, an equivalent expression is

(42) $$\Phi(\bar{B}, \bar{d}) = \min_{0 \le d_1' \le W} \{T|d_1 - d_1'| + H(B_1 + B_2) + \frac{d_1'\ H}{1 - \alpha} + \alpha \int_0^\infty g^{d_1'}(m_1|B_1)\ \Phi(m_1, B_2, \bar{d}')\ dm_1\}.$$

Furthermore, the minimization process is independent of the value for B_2. Thus the dependence on B_2 can be dropped and (42) becomes,

$$(43) \qquad \Phi(B_1, \bar{d}) = \min_{0 \le d_1' \le W} \{T|d_1 - d_1'| + H(B_1) + \frac{d_1' H}{1 - \alpha}$$

$$+ \alpha \int_0^\infty g^{d_1'} (m_1|B_1)\ \Phi(m_1, \bar{d}')\ dm_1\}.$$

Finally note that $\bar{d}$ can be completely described by a single vector element d_1 (since $d_2 = W - d_1$) and of course the same is true for $\bar{d}'$.

Thus, in the limit the two shop minimization problem may be reduced to a single shop formulation where the costs associated with the system can be thought of as:

- a cost of turning a worker on or off $(T|d_1 - d_1'|)$
- a backlog holding cost HB_1
- and a cost of employing a server $d_1'(\frac{H}{1 - \alpha})$.

In this "imaginary system" control is exercised by opening or closing a worker's station. Note that what the conversion to a single shop formulation does is translate the holding cost for backlog in shop 2 into a cost of employing a server in shop 1.

Q.E.D.

Property 5:

One property of the optimal control system which at first seems obvious, is as follows:

At any stage, (for a given number of workers at the previous stage) the optimal number of workers in shop i either remains the same or increases as the backlog in shop i is increased and backlog in the other shop is held

constant, this conjectured property will hereafter be called the "increasing" property. Conversations with other researchers have revealed that this is a topic of considerable interest.

Despite its intuitive appeal, the increasing property for single shop systems does not always hold. In Haas [12] a counter example is given for a non stationary single shop system with transfer costs. Winston [25] proved that the increasing property does hold for a one shop (queue) stationary system which includes a control option of 0, 1, or 2 workers (servers) and transfer costs.

For stationary undiscounted parallel shop systems our computational experience has shown that the increasing property rarely, if ever, fails to hold. However, for the discounted system, it is possible to find cases where the increasing property does not always hold (see discussion in next section). For series shops, counter examples to the increasing property are abundant even for undiscounted systems (see Part 2: Series Shops). If there are shop (queueing) systems where the optimal control rules can be shown to exhibit the "increasing property," we suspect that the systems are parallel (as opposed to series) systems and that discounting is not used in the cost structure of the system.

Geometric Interpretation and Computational Efficiencies

With these properties established for the 2-shop infinite and finite horizon case, it is possible to indicate geometrically the nature of the optimal control structure. Figure 3a shows an example structure for the case where the manpower allocation (5 workers in the system) for the previous stage was $\bar{d} = (3, 2)$ and where $T > H(R_2 - R_1)\ \alpha/(1 - \alpha)^2$. The rectangular region with one corner at the origin is the "do nothing" region required by property 2. The other "do nothing" region is a consequence of property 1. At relatively large values of B_1 and B_2, the position of the isopols can be determined by the one shop formulation described in property 4. For heavily discounted systems (on the order of

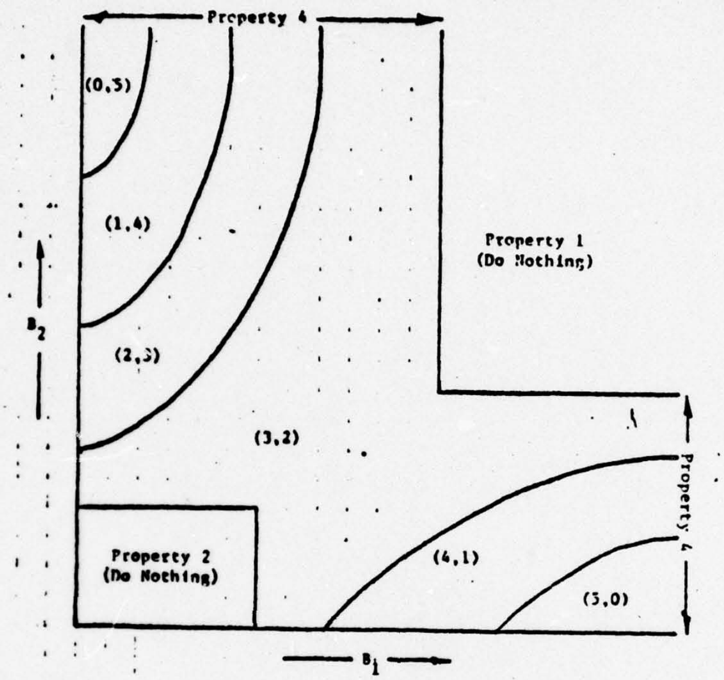

Figure 3a: Graphical Representation of Optimal Control Structure for 2-shop, 5 worker System. When the Previous Policy is (3, 2) and $T \geq H(R_2 - R_1)\alpha/(1 - \alpha)^2$.

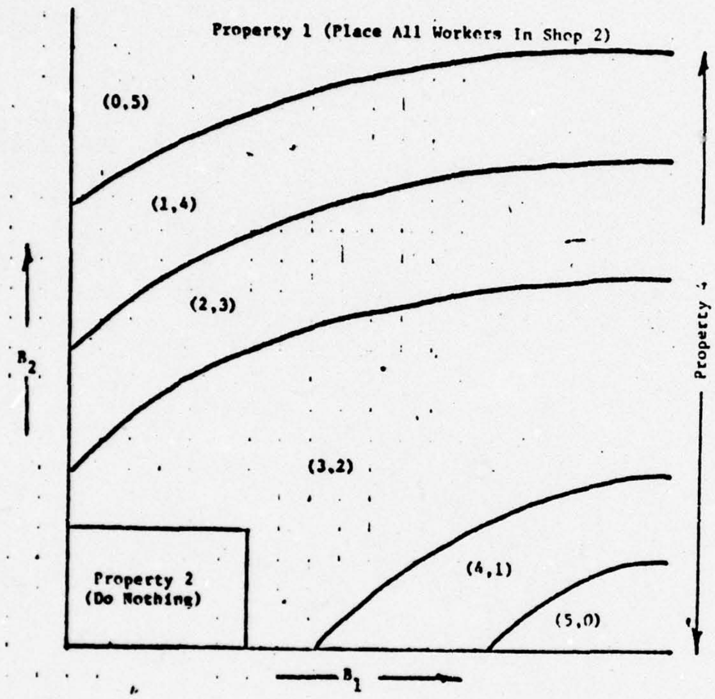

Figure 3b: Graphical representation of optimal control structure for 2-shop, 5-worker system when the previous policy is (3, 2) and $T < H(R_2 - R_1)\alpha/(1 - \alpha)^2$.

10% per period or week), the position of the isopols changes (see dotted line in Figure 3a) revealing situations where the increasing property does not hold. However, for realistic discount rates (say 20% per year or less) we have found little change from the undiscounted systems for the optimal policy set.

When $T < H(R_2 - R_1)\ \alpha/(1-\alpha)^2$ the optimal control structure takes on a different form. In this case, property 1 no longer requires the "do nothing" region shown in figure 3a. Instead, property 1 requires that for shop 2, whenever the backlog in what shop exceeds a given level, the optimal policy is to place all workers in that shop. The optimal control structure for such a system is shown in figure 3b.

In figure 4 the consequences of the inertia property for a 3 worker system are noted. The four planes shown give examples of isopol configuration when the number of workers in shop 1 at the previous stage are 3 for the top plane, 2 for the plane below, 1 for the plane below that, and 0 for the bottom plane. Note that in those regions of the state space where the optimal policy is to increase the number of workers in shop 1 isopols correspond to those in the bottom plane. When the optimal policy is to decrease the number of workers in shop 1, isopols correspond to those in the top plane.

To see why, consider the following example: Suppose that the optimal policy at point $(\bar{B}, 3, 0)$ is $(1, 2)$. Then points $(\bar{B}, 2, 1)$ and $(\bar{B}, 1, 2)$ fulfill the first requirement of Theorem 3 (i.e., $d_1 \geq d_1' \geq p_1$). Now if the optimal policy in any of the three points is to decrease the number of workers in shop 1, the second condition of Theorem 3 ($d_1' \geq p_1'$) is satisfied and p_1 must equal p_1'. This does not yet guarantee that $p_1 = p_1'$. The only way to establish this is to show that the second condition of Theorem 3 is indeed satisfied.

Theorem 4 can be used to do this. Suppose that $d_1' < p_1'$ then $p_1 < d_1' < p_1'$. But Theorem 4 stipulates that $p_1 \geq p_1'$, a contradiction. Thus, $d_1' \geq p_1'$ and

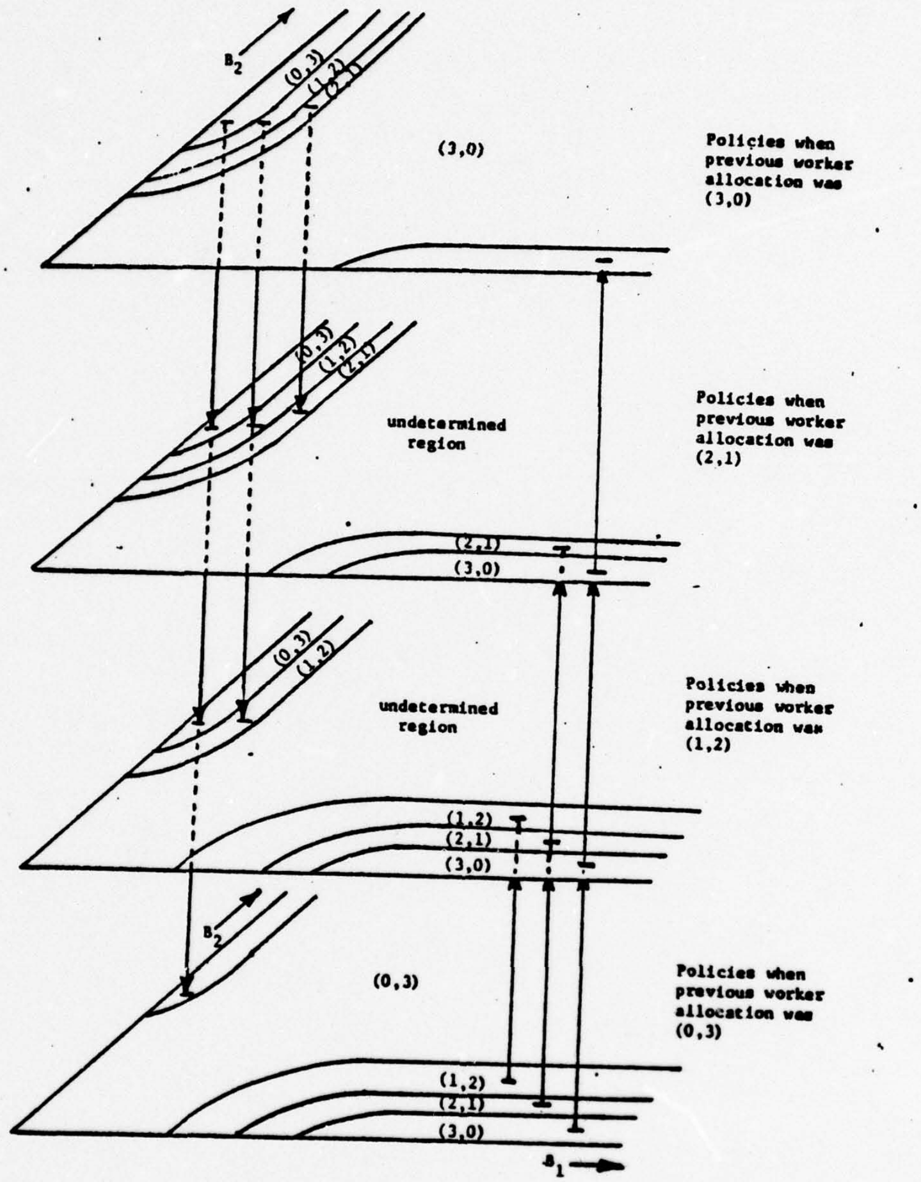

Figure 4: The Effect of Inertia on a Two-Shop Three Worker Parallel System

must equal p_1'. To summarize, if $\bar{p}$ is optimal at $(\bar{B}, 3, 0)$, and if $3 \geq d_1' \geq p_1$, then $\bar{p}$ is optimal at $(\bar{B}, d_1', d_2')$. Similar logic shows that if $\bar{p}$ is optimal at $(\bar{B}, 0, 3)$ and if $0 \leq d_1' \leq p_1$, then $\bar{p}$ is optimal at $(\bar{B}, d_1', d_2')$.

The "undetermined" regions are undetermined in the sense that the inertia property itself cannot be used to determine the policies in the regions. Properties 1 and 2 can be used to fix portions of the "undetermined" regions. In addition, it is easy to see that if the increasing property holds, the "undetermined" regions are do nothing (i.e., maintain the present worker allocation) regions. Assuming that the increasing property holds (when the shop system is modeled as an undiscounted stationary process) an efficient algorithmic approach can be used to determine optimal strategies.

Consider a 2-shop W-worker system where the objective is to minimize the total discounted operating cost over an infinite horizon. Applying the well known Howard [13] policy iteration technique in a straight-forward fashion requires at each iteration, for each state, a search over all possible policies. Assuming (for the moment) the existence of the increasing property a policy iteration may be completed in the following manner. Determine policies for the states

$$(\bar{B}, 0, W), (\bar{B}, W, 0), \bar{B} \geq 0.$$

Policies for the states

$$(\bar{B}, 1, W-1), (\bar{B}, 2, W-2), \ldots, (\bar{B}, W-1, 1) \quad \bar{B} \geq 0$$

are determined directly by applying the inertia and increasing properties. Use of this property reduces computational effort from one which is quadratically expanding in W to one which is linearly expanding. In addition, further economies can be effected. Starting the iteration at states $(\bar{0}, 0, W)$ and $(\bar{0}, W, 0)$ and moving to states with successively larger backlogs the increasing property can be used to limit the policy search for the $(\bar{B}, 0, W)$ and $(\bar{B}, W, 0)$ states.

Once the algorithm converges, one additional iteration checking for potential optimal (decreasing) policies insures the existence of the increasing property. It should be noted parenthetically (recall the earlier discussion of Property 5) that instances where the property does not hold (for the stationary undiscounted system) are rare, and possibly non-existent.

It is possible to quantify precisely the inertia property's ability to reduce the size of a problem. Suppose that a two shop system has a limited state space where

$$0 \leq B_1 \leq 1000 \text{ mhrs. and } 0 \leq B_2 \leq 1000 \text{ mhrs.}$$

(Of course the problem as formulated assumes that the backlog may take on any value no matter how large. However, it is not unreasonable to place an upper bound on allowable backlog thereby making a computational solution possible.) Suppose, further, that for computational purposes the state space has been divided into 40x40 manhour blocks. For a fixed previous stage worker allocation there will, therefore, by 25x25 or 625 states associated with that allocation alone. If, for example, there are 30 total workers in the system, there will then by 625x30 or 18,750 total states. But by utilizing the inertia property the problem can be reduced to one where only 625x2 or 1,250 states must undergo policy search.

It should be noted that, for computational purposes, it does not appear necessary to be overly concerned with the existence of the increasing property. Our experience has been that the inertia property, and Property 2 account for as much as 95% or more of the policies in the interior states.

Extensions of Results to N-Shop Systems

Extensions of the results presented here to N-shop systems are generally straightforward. The reader is referred to Haas [12] which contains a complete discussion of the extension to N shops.

Summary

To summarize, we have shown that parts of the state space for the two shop optimal control problem can be sudivided into regions where either; the optimal control policy is known exactly (as with properties 1 and 2), or where the process for finding a solution is of reduced complexity (as with property 4). Furthermore Property 3, by showing that all optimal policies are completely determined when optimal policies are known for only ($\bar{B}$, W, 0) and ($\bar{B}$, 0, W), drastically reduces computer storage and time requirements when solution of a problem by computational methods is necessary. This information should prove to be more and more useful as production facilities move to higher degrees of systematic or automated control.

Even if policies are not determined precisely, a general knowledge of the layout of the control structure can be used to advantage. The two fundamentally different forms (described by Property 1) which the optimal control structure may take are a case in point. By utilizing the information in Property 1, the shop system controller can determine from shop parameters which basic form of control to apply to a system, and then act accordingly. Such insights into control structure can be very valuable when exercising shop control.

PART II

Introduction

Before beginning this discussion of the formulation of series shop systems the reader is referred to the introductory and formulation discussion in Part 1 (parallel shops).

The formulation for the series shop system is quite similar to that for the parallel system. In fact, the cost structure for the series system is identical to the parallel system. The only difference in the two is the flow of work from shop to shop. In the series system, work which exists one shop enters the next shop in the series as input where it awaits processing in that shop (see Figure 1). It is assumed that work completed in shop 1 during the period is not available to shop 2 until the beginning of the next period. There are no backlog limitations for any shop, and input to the first shop is the only stochastic element of the system.

The difference in workflow (parallel vs. series) can be completely accomodated by the transition function $f^{\bar{P}}(\bar{m}|\bar{B})$ so that the recursive equation for the series system appears exactly the same as for the parallel system. That is:

$$\Phi(\bar{B}, \bar{d}) = \min_{0 \leq p_1 \leq W} \{T|p_1 - d_1| + H \sum_{i=1}^{2} B_i + \alpha \int_0^{\infty} f^{\bar{P}}(\bar{m}|\bar{B})\ \Phi(\bar{m}, \bar{p})d\bar{m}\}.$$

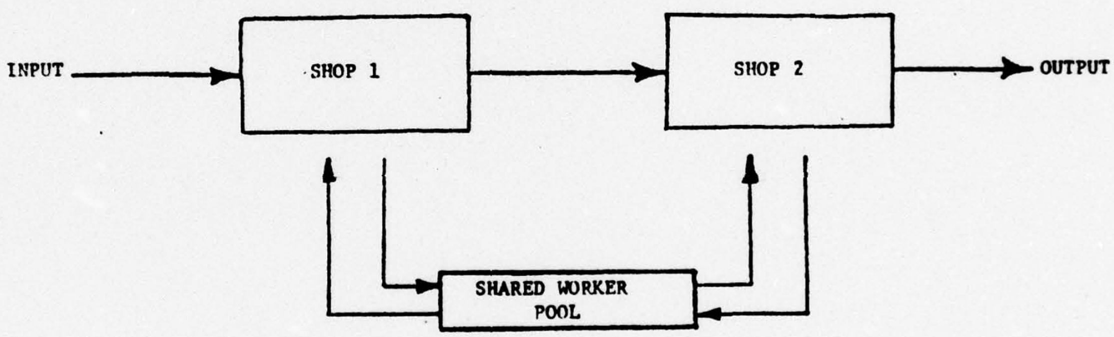

Figure 1. Two Shops in Series

Note that the only difference in the parallel system formulation is the implicit fact that, $\bar{f}^P(\bar{m}|\bar{B})$ now reflects the workflow movement of a series system. Otherwise, the symbolism used above, and throughout Part 2, is the same as that defined in Part 1.

The Structure of the Optimal Control Policy

In this section three fundamental properties of the optimal control policies for an infinite horizon system will be determined. These properties will be used to reveal the basic nature of the optimal control structure for series system.

Property 1:

The first property to be established shows that if the transfer cost is less than a specified critical number, then when backlog is sufficiently great, the optimal policy is to reallocate manpower so that all workers are in shop 2. If T is greater than the critical number, it is never optimal, no matter what the backlog level, to transfer workers.

Before beginning the proof for property 1, two necessary lemmas will be established.

Lemma 1:

$$\Phi(B_1 + \Theta, B_2, \bar{d}) - \Phi(B_1, B_2, \bar{d}) \leq \frac{\Theta H}{1 - \alpha} \tag{1}$$

(The same property holds in B_2.)

Proof:

The proof follows that for Lemma 1 in Part 1. (As in Part 1, it is easy to show that in the limit as $B_i \to \infty$, the inequality of Lemma 1 becomes an equality.)

Lemma 2:

$$|\Phi(\bar{B}, \bar{d}) - \Phi(B, \bar{d}')| \leq |d_1 - d_1'|\ T$$

Proof:

The proof follows that for Lemma 2 in Part 1.

Theorem 1.

Let $R = \text{Max}\ \{R_1, R_2\}$. (Note that R_i is the period man hour capacity of shop i)

Define

$$n = n_1 + n_2.$$

1. When n_1 can be selected large enough that

$$T \leq \frac{HR_2 \sum\limits_{i=1}^{n_1-1} i\alpha^i - \dfrac{\alpha^{n_1}(2n_1RH)}{1 - \alpha}}{1 + \alpha^{n_1}}, \tag{2a}$$

and if n_2 is selected so that

$$\frac{2RH\alpha^{n_2+1}}{(1 - \alpha)^2} \leq T \tag{2b}$$

and so that,

$$T(1 - \alpha) \leq HR_2 \sum_{i=n_1}^{n_1+n_2-1} \alpha^i - \frac{\alpha^{n_1+n_2}2RH}{1 - \alpha} \tag{2c}$$

Then, when $B_2 \geq n \cdot R_2 \cdot W$, *it is optimal to place all workers in shop 2 for the first n_1 stages.* (In other words, if the backlog in shop 2 is big enough, and

$$T < HR_2\alpha/(1 - \alpha)^2 ,$$

then it is optimal to place all workers in shop 2.)

2. When

$$T > HR_2\alpha/(1 - \alpha)^2 \tag{2d}$$

it is always optimal, no matter what the backlog level, to hold the worker allocation of the previous stage constant.

Proof:

It will first be assumed that it is never optimal in the first n_1 stages to decrease the number of workers in shop 2. Further, it will be assumed that if it is optimal (in the first n_1 stages) to increase workers in shop 2, that increase must take place at stage 1. These assumptions leave only two candidate optimal policies, either maintain the worker allocation of the previous stage, or increase (at stage 1) the number of workers in shop 2. The first two parts of the proof will involve a comparison of these candidate policies.

In part 1 it will be shown that (given the above assumptions) and given that (2a) holds it is optimal to increase the number of workers in shop N. In fact, it will be shown that it is optimal to place all workers in shop 2. In part 2 it will be shown that given that (2d) holds it is always optimal to maintain the worker allocation of the previous stage. In part 3 the proof will be completed by showing that each of the initial assumptions must hold.

Part 1: Assume that inequality (2a) holds. Now fix an input realization and define option 1 as follows: For the first n_1 stages, policy $\bar{d}'$ ($d_2' \neq 0$) is in effect. Thereafter, the system is optimally controlled. The expected cost of the system is

$$\Phi'(\bar{B}, \bar{d}) = C_{n_1}^{t'} + C_{n_1}^{h'} + \alpha^{n_1} C'_{n_1+1} \quad ,$$

where $C_{n_1}^{t'}$ is the expected discounted transition cost through n_1 transitions,

$C^{h'}_{n_1}$ is the expected discounted holding cost through n_1 transitions, and

C'_{n_1+1} is the expected discounted cost incurred for all stages after n_1.

For the same input realization define option 2 as follows: The worker allocation for the first n_1 stages is $\bar{d}''$. $\bar{d}''$ is the same as $\bar{d}'$ except that $d''_2 = d'_2 - 1$ and the worker lost to shop 2 is added to shop 1.

This policy remains in effect for the first n_1 stages. Thereafter, the system is optimally controlled. The expected cost for option 2 is

$$\Phi''(\bar{B}, \bar{d}) = C^{t''}_{n_1} + C^{h''}_{n_1} + \alpha^{n_1} C''_{n_1+1} \quad .$$

For any input realization, the backlog levels for option 1 at stage $n_1 + 1$ and for option 2 at stage $n_1 + 1$ are limited in separation. The limitation is as follows: If,

B'_i = backlog in shop i for option 1 at stage n_1+1,

B''_i = backlog in shop i for option 2 at stage $n_1 + 1$,

then,

$$|B'_i - B''_i| \le 2n_1 R \quad .$$

From Lemma 1 it can easily be shown that,

$$|\Phi(\bar{B}'', \bar{d}') - \Phi(\bar{B}'', \bar{d}'')| \le |d'_1 - d''_1| T = T \quad .$$

Then,

$$\begin{aligned} |C'_{n_1+1} - C''_{n_1+1}| = |\Phi(\bar{B}', \bar{d}') - \Phi(\bar{B}'', \bar{d}'')| &\le |\Phi(\bar{B}', \bar{d}') - \Phi(\bar{B}'', \bar{d}')| \\ &+ |\Phi(\bar{B}'', \bar{d}') - \Phi(\bar{B}'', \bar{d}'')| \\ &\le \frac{2n_1 RH}{1-\alpha} + T. \qquad (3) \end{aligned}$$

Removing the absolute value sign, multiplying by α^{n_1} and rearranging yields,

$$\alpha^{n_1} C''_{n_1+1} \geq \alpha^{n_1} C'_{n_1+1} - \alpha^{n_1}\left(\frac{2n_1 RH}{1-\alpha} + T\right) \quad . \tag{4}$$

Because shop 2 cannot run out of work for n_1 stages,

$$C^{h''}_{n_1} = C^{h'}_{n_1} + HR_2 \sum_{i=1}^{n_1-1} i\alpha^i \quad . \tag{5}$$

Adding (4) and (5),

$$C^{h''}_{n_1} + \alpha^{n_1} C''_{n_1+1} \geq C^{h'}_{n_1} + \alpha^{n_1} C'_{n_1+1} - \alpha^{n_1}\left(\frac{2n_1 RH}{1-\alpha} + T\right) + HR_2 \sum_{i=1}^{n_1-1} i\alpha^i . \tag{6}$$

At this point it has been assumed that it is never optimal to decrease the number of workers in shop 2. Thus, it is only necessary to consider the case where selection of policy $\bar{d}'$ results in an increase in the number of workers in shop 2. Therefore,

$$C^{t''}_{n_1} = C^{t'}_{n_1} - T. \tag{7}$$

Adding (6) and (7) yields,

$$C^{h''}_{n_1} + C^{t''}_{n_1} + \alpha^{n_1} C''_{n_1+1} \geq C^{h'}_{n_1} + C^{t'}_{n_1} + \alpha^{n_1} C'_{n_1+1} - \alpha^{n_1}\left(\frac{2n_1 RH}{1-\alpha} + T\right) + HR_2 \sum_{i=1}^{n_1-1} i\alpha^i - T. \tag{8}$$

Now recall inequality (2a)

$$T \leq \frac{HR_2 \sum_{i=1}^{n_1-1} i\alpha^i - \dfrac{\alpha^{n_1}(2n_1 RH)}{1-\alpha}}{1 + \alpha^{n_1}} \tag{2a}$$

Multiplying by the denominator gives,

$$T(1 + \alpha^{n_1}) \le HR_2 \sum_{i=1}^{n_1-1} i\alpha^i - \frac{\alpha^{n_1}(2n_1RH)}{1-\alpha} .$$

Rearranging yields,

$$\alpha^{n_1}\left(\frac{2n_1RH}{1-\alpha} + T\right) \le HR_2 \sum_{i=1}^{n_1-1} i\alpha^i - T. \qquad (9)$$

(8) and (9) imply that,

$$C_{n_1}^{h''} + C_{n_1}^{t''} + \alpha^{n_1} C_{n_1}^{h''} \ge C_{n_1}^{h'} + C_{n_1}^{t'} + \alpha^{n_1} C'_{n_1+1}. \qquad (10)$$

Eq. (10) shows that <u>if</u> the policy selected at stage 1 is maintained for the first n_1 stages, and if (2a) holds policy $\bar{d}'$ provides a lower expected cost than policy $\bar{d}''$. Repeated application of this result shows that <u>if</u> the policy selected at stage 1 is maintained for the first n_1 stages, and if (2a) holds, the expected return when all W workers are placed in shop 2 at stage 1 is less than or equal to the expected return with any other policy selected.

<u>Part 2</u>: Note that $HR_2 \sum_{i=1}^{\infty} i\alpha^i$ represents an upper bound on holding cost reduction which can result from the transfer of a single worker. Clearly, then, if

$$T > HR_2 \sum_{i=1}^{\infty} i\alpha^i = HR_2\alpha/(1-\alpha)^2,$$

it is never optimal to make a worker transfer.

<u>Part 3</u>: The validity of the initial assumptions will now be established. This is done by comparing the assumption with the alternative policy, using the relationships in the theorem statement.

<u>Assumption</u>: Any policy change at stage 2 through n_1 which increases the number of workers in shop 2 is nonoptimal.

Fix an input realization. Suppose that stage X+1, $2 \le X+1 \le n_1$ is the first point where the optimal policy is to increase the number of workers in shop 2, and suppose further that the shops are optimally controlled thereafter. Define this as option 1. Define option 2 as follows:

Option 2 is the same as option 1 except that the increase in shop 2 is made one stage earlier, at stage X.

Since shop 2 cannot run out of work,

$$C^{h''}_{n_1+n_2} = C^{h'}_{n_1+n_2} - HR_2 \sum_{i=X}^{n_1+n_2-1} \alpha^i \tag{11}$$

The difference in transfer costs for the two options can be expressed by

$$C^{t''}_{n_1+n_2} \le C^{t'}_{n_1+n_2} + T\alpha^{X-1}(1-\alpha). \tag{12}$$

By arguments similar to those used earlier, it is easily seen that

$$\alpha^{n_1+n_2} C''_{n_1+n_2+1} \le \alpha^{n_1+n_2} C'_{n_1+n_2+1} + \frac{\alpha^{n_1+n_2} 2RH}{1-\alpha}. \tag{13}$$

Adding (11), (12), and (13) gives

$$\begin{aligned} C^{h''}_{n_1+n_2} + C^{t''}_{n_1+n_2} + \alpha^{n_1+n_2} C''_{n_1+n_2+1} \le\ & C^{h'}_{n_1+n_2} + C^{t'}_{n_1+n_2} \\ & + \alpha^{n_1+n_2} C'_{n_1+n_2+1} + \frac{\alpha^{n_1+n_2} 2RH}{1-\alpha} \\ & + T\alpha^{X-1}(1-\alpha) - HR_2 \sum_{i=X}^{n_1+n_2-1} \alpha^i . \end{aligned} \tag{14}$$

Since X is never larger than n_1, inequality (2c) implies that

$$\frac{\alpha^{n_1+n_2} 2RH}{1-\alpha} + T\alpha^{X-1}(1-\alpha) \le HR_2 \sum_{i=X}^{n_1+n_2-1} \alpha^i. \tag{15}$$

(14) and (15) imply that

$$C^{h''}_{n_1+n_2} + C^{t''}_{n_1+n_2} + \alpha^{n_1+n_2} C''_{n_1+n_2+1} \leq C^{h'}_{n_1+n_2} + C^{t'}_{n_1+n_2} + \alpha^{n_1+n_2} C'_{n_1+n_2+1}$$

<u>Assumption</u>: Any policy change at stages 1 through n_1 which decreases the number of workers in shop 2 is nonoptimal.

Fix an input realization.

Define option 1 as follows: At stage X policy $\bar{d}'$ is in effect. Policy $\bar{d}'$ decreases the number of workers which were in shop 2 at the previous stage and increases the number of workers in shop 1. Option 2 is defined as follows: At stage X the policy in force is $\bar{d}'' = (d_1' - 1, d_2' + 1)$.

Since shop 2 cannot run out of work,

$$C^{h''}_{n_1+n_2} = C^{h'}_{n_1+n_2} - HR_2 \sum_{i=X}^{n_1+n_2-1} \alpha^i. \tag{16}$$

Difference in transfer costs for the two options

$$C^{t''}_{n_1+n_2} \leq C^{t'}_{n_1+n_2} - T\alpha^{X-1}(1-\alpha) \tag{17}$$

By arguments similar to those used earlier, it is easily seen that

$$\alpha^{n_1+n_2} C''_{n_1+n_2+1} \leq \alpha^{n_1+n_2} C'_{n_1+n_2+1} + \frac{\alpha^{n_1+n_2} 2RH}{1-\alpha}. \tag{18}$$

Adding (16), (17), and (18) together and using (2c), the same logic used in the previous assumption shows that

$$C^{h''}_{n_1+n_2} + C^{t''}_{n_1+n_2} + \alpha^{n_1+n_2} C''_{n_1+n_2+1} \leq C^{h'}_{n_1+n_2} + C^{t'}_{n_1+n_2} + \alpha^{n_1+n_2} C''_{n_1+n_2+1}$$

Q.E.D

Property 2:

Just as backlog can be so large that the optimal policy is specified, backlog levels also may be so low that the optimal policy is also specified.

Before proving this property, one lemma must be established.

Lemma 3:

$$\Phi(\bar{B}, \bar{d}) \leq \Phi(B_1 + \theta_1, B_2 - \theta_1 + \theta_2, \bar{d}), \ 0 \leq \theta_i \leq B_i, \ i = 1, 2.$$

Proof: Define

$$\bar{\mu} = (\bar{B}, \bar{d})$$

$$\bar{\mu}' = (B_1 + \theta_1, B_2 - \theta_1 + \theta_2, \bar{d})$$

It is easy to show that for any given input realization, if the optimal policies generated for starting at point $\bar{\mu}'$ are used, the total backlog ($\sum B_i$) at any stage of the sequence starting at point $\bar{\mu}$ will always be the same or less than the total backlog level generated from starting at point $\bar{\mu}'$.

Thus, the total expected cost of operating the system for any input realization must be more if the system is presently at $\bar{\mu}'$.

Q.E.D.

With this lemma established, it is possible to proceed with the following theorem:

Theorem 2:

If the state of the system $(\bar{B}, \bar{d})$ is such that $B_i \leq d_i R_i \ \forall \ i$, the optimal policy is to maintain the present allocation $\bar{d}$.

Proof:

Consider what happens to the backlog levels at a point in time immediately before the upcoming input surge. Assume that with policy $\bar{d}$ the backlog level would have been $\bar{B}$ immediately before the input surge.

If a worker from shop 1 is removed, the net effect is to increase the backlog in a shop 1 by $\theta_1 (0 \leq \partial_1 \leq R_1)$ and to decrease the work in shop 2 by θ_1. If a worker is removed from shop 2 the backlog in shop 2 is increased by $\theta_2 (0 \leq \theta_2 \leq R_2)$ but shop 1's backlog is not affected. The backlog in the shop which received the removed worker will remain unchanged since, by the theorem assumption, each shop has enough workers to complete all work which was in the shop at the previous transition.

Thus, for any policy $\bar{p}$ which is different than $\bar{d}$, the backlog immediately before the upcoming input surge will take the form:

$$(B_1 + \theta_1, B_2 - \theta_1 + \theta_2).$$

Thus, from Lemma 3, it is easy to see that

$$\int_0^\infty \Phi\ (\bar{m}, \bar{d})\ f^{\bar{p}}(\bar{m}|\bar{B})\ d\bar{m} \geq \int_0^\infty \Phi\ (\bar{m}, \bar{d})\ f^{\bar{d}}(\bar{m}|\bar{B})\ d\bar{m}.$$

Since policy $\bar{p}$ also entails transfer costs, it follows that it is cheaper to maintain the present policy $\bar{d}$.

Q.E.D.

Property 3:

Property 3 is established for the series system in exactly the same way as for the parallel system. Theorems 3 and 4 of Part 1, which established property 3 for parallel shops, are restated here.

Theorem 3:

Consider two distinct points in the shop state space $(\bar{B}, \bar{d})$ and $(\bar{B}, \bar{d}')$.

Let p be optimal at $(\bar{B}, \bar{d})$ and let $\bar{p}'$ be optimal at $(\bar{B}, \bar{d}')$. Furthermore, for all i, let either

$$d_i \leq d_i' \leq p_i \text{ and } d_i' \leq p_i'$$

or $$d_i \geq d_i' \leq p_i \text{ and } d_i' \geq p_i'$$

Then p' is also optimal at $(\bar{B}, \bar{d})$, and $\bar{p}$ is also optimal at $(\bar{B}, \bar{d}')$.

Proof:

The proof is identical to that for theorem 3 in Part 1.

Theorem 4:

Fix some point $\bar{B}$ in backlog. Let $\bar{p}$ be the optimal policy when the worker allocation for the previous transition is $\bar{S}$. Let $\bar{p}'$ be the optimal policy at $\bar{B}$ when the worker allocation for the previous transition is $\bar{S} + \bar{Q}$, and where $\bar{Q}$ is a vector whose elements may be positive or negative. Furthermore, let

$$S_i < S_i + Q_i < p_i \ \forall\ i \text{ such that } S_i < p_i$$

$$S_i > S_i + Q_i > p_i \ \forall\ i \text{ such that } p_i < S_i$$

Then, for all i

$$S_i + Q_i \leq p_i' \text{ when } S_i < p_i, \text{ and}$$

$$S_i + Q_i \geq p_i' \text{ when } S_i > p_i.$$

Proof:

The proof is identical to that for theorem 4 in Part 1.

Geometric Interpretation:

The properties developed yield a great deal of information about the optimal control structure of a series shop system. Figure 2 depicts the general structure of the optimal control policies. Property 1 indicates what optimal policy will be in force in shop 2 when backlog in that shop becomes large. When

$$T < HR_2\alpha/(1-\alpha)^2,$$

as it is for the system represented in Figure 2. That policy will be to place all workers in shop 2. If T is larger, the problem is trivial. (No worker is ever transferred.)

Property 2 indicates that in an area close to the origin, the optimal policy is to maintain the policy of the previous stage.

Property 4 shows that in a region where backlog in shop 1 is large the optimal control structure can be found by solving a 1 shop problem.

The effect of property 3 on the series shop optimal control structure is shown in Figure 3. The four planes shown give examples of isopol configuration when the number of workers in shop 1 at the previous stage are 3 for the top plane, 2 for the plane below, 1 for the plane below that, and 0 for the bottom plane. Note that in those regions of the state space, where the optimal policy is to increase the number of workers in shop 1, isopols correspond to those in the bottom plane. When the optimal policy is to decrease the number of workers in shop 1, the isopols correspond to those in the top plane.

Note that the above properties can be used in the same manner as those for parallel systems to build efficient computational solution methods. (See the discussion in Part 1.)

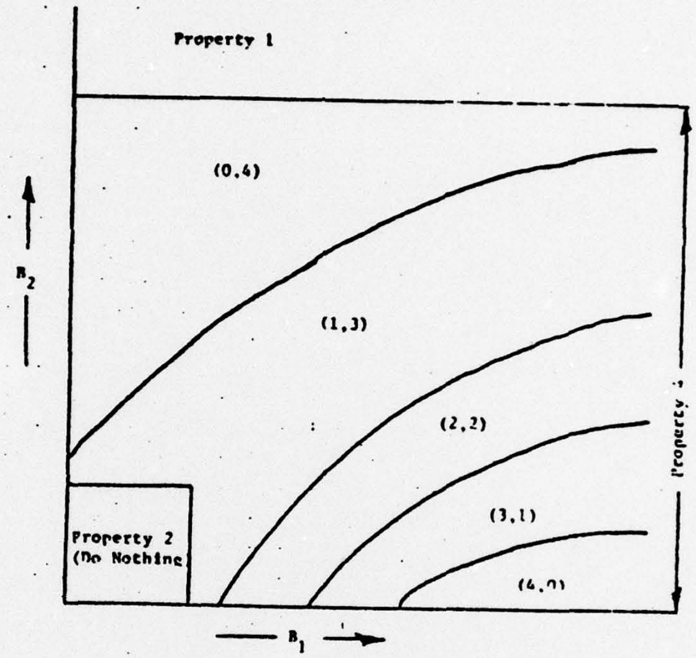

Figure 2: General structure of control policies for a 2-shops in series system (4 workers) where the previous policy is (1, 3).

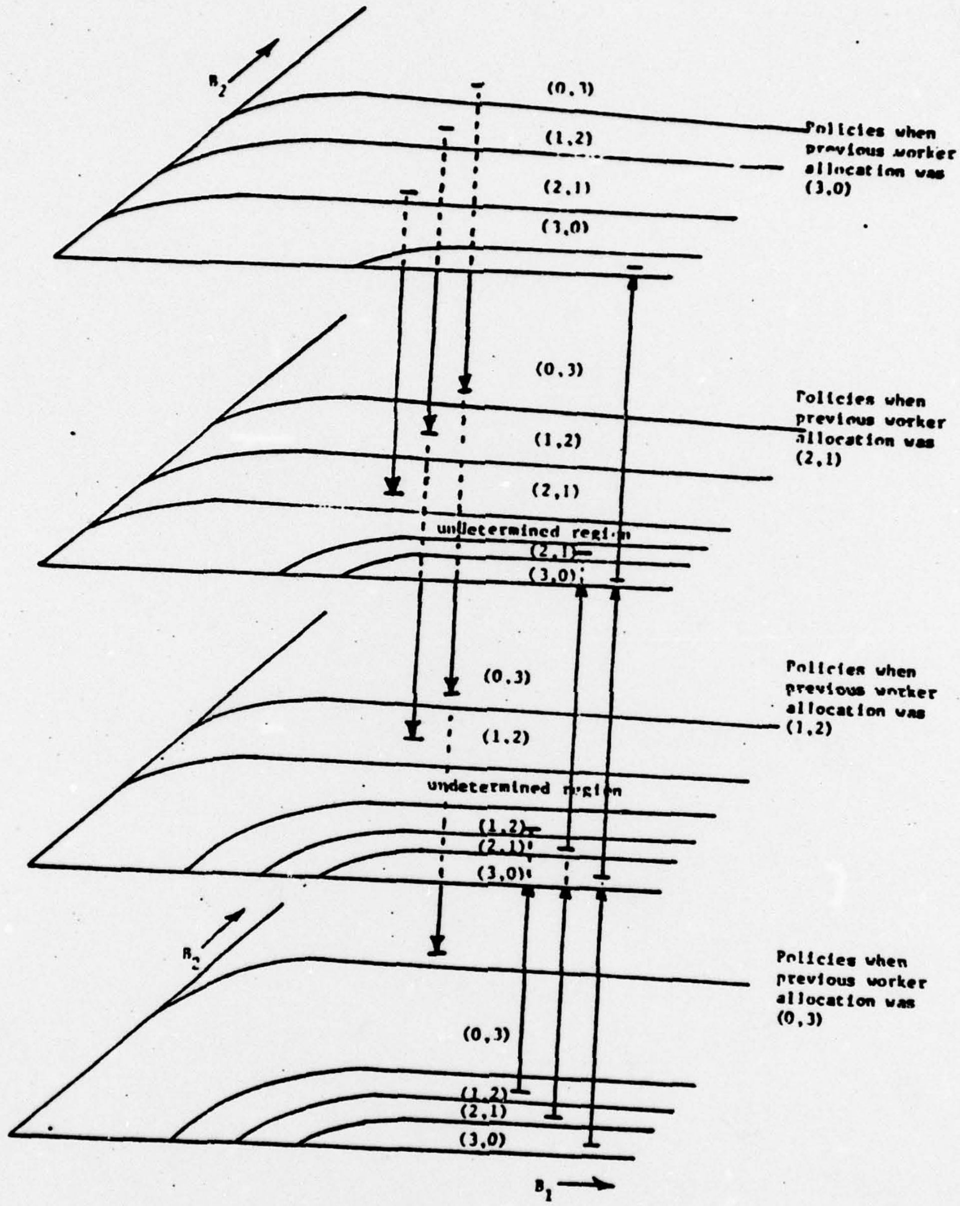

Figure 3: The Effect of Inertia on a Two-Shop, Three Worker, Series System

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